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THE STABILITY OF THE STEADY MOTIONS OF A SYMMETRICAL GYROSTAT ON AN ABSOLUTELY ROUGH HORIZONTAL PLANE†

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The problem of the motion, without slipping, of a dynamically symmetrical gyrostat on a fixed horizontal plane is investigated. It is shown that Chaplygin's equations express, in projections on the axes of a semimobile system of coordinates, the theorem on the angular momentum of the gyrostat about the point of contact with the plane. All possible steady motions of a heavy symmetrical gyrostat on an absolutely rough plane including the case when the nutation angle is equal to zero are investigated using the generalized Routh theorem. The effect of the rotor on the stability of steady motions is also examined. The case when the body of the gyroscope is a solid with a circular base, in particular, a disc with a rotor, is considered. It is shown that the rotating rotor has a stabilizing influence on equilibrium of the gyrostat and a destabilizing influence on the rolling of the gyrostat about a straight line, provided the axial angular momentum of the gyrostat is zero.

Chaplygin derived the equations of motion of a non-holonomic system in generalized coordinates for systems called Chaplygin systems [1]. Later, by solving the problem of the motion without slipping of a heavy symmetrical gyrostat – a solid of revolution with a rotor, he pointed out a numer of special cases when the integration of a equations of motion is reduced to quadratures, and he made a number of observations on the nature of the motion of a gyrostat under these conditions. Below we investigate the stability of these steady motions.

In [2] the stability of the rolling of a disc with a rotor on an absolutely rough plane was investigated, and the necessary and sufficient condition for the stability of the permanent rotations of the gyrostat about the axis of symmetry was also obtained [3]. The necessary and sufficient condition for the stability of the steady motions of a gyrostat, assuming that the angle of nutation is non-zero, was derived in [4]. When investigating the stability by the direct Lyapunov method, the representation of the unknown first integrals in the form of hypergeometric series was essentially used.

The motion on an absolutely rough plane of a heady body, the mass distribution and surface shape of which are arbitrary, was investigated in [6, 7], and also some special cases of bodies in [5, 8, 9]. The stability of the steady motions around the vertical of a heavy gyrostat of arbitrary shape was investigated in [10]. A theory of the motion of bodies, in particular, on an absolutely rough surface, was proposed in [11]. The conditions for the stability of all steady motions of a solid of revolution, with the exception of rotation around an axis of symmetry, were obtained in [12].

Below we extend the investigation of the dynamics of a system on an absolutely rough plane. An extension of Routh's theorem to investigate the stability of the steady motions of conservative non-holonomic Chaplygin systems proposed in [13] enables us to investigate completely the stability of the steady motions of a gyrostat and to generalize a number of results obtained earlier.

1. SOME GEOMETRICAL AND KINEMATIC FORMULAE

Consider the motion, without slipping, on a horizontal plane of a heavy rigid body, bounded by a convex surface of revolution, where the axis of symmetry of the surface of the body coincides with its axis of dynamic symmetry. We will connect to the body a rotor which rotates around the axis of symmetry of the body. The rigid body, together with the rotor, form a gyrostat – a mechanical system with moving parts, the distribution of the mass m of which remains unchanged [14].

Suppose $O'xyz$ is a fixed right system of coordinates with origin O' and with the x and y axes in the reference plane and the z axis directed upwards; the system of axes $G\xi\eta\zeta$ is rigidly connected with the gyrostat, its origin is at the centre of mass G of the gyrostat, the ζ axis is directed upwards with respect to the axis of symmetry and the $\xi\eta$ axes are directed along the principal central axes of inertia of the

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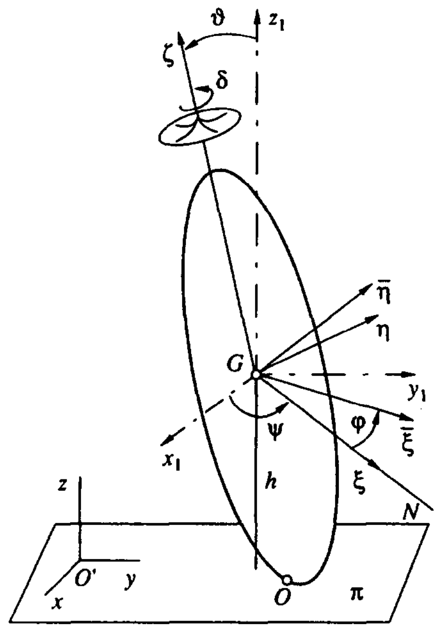

Fig. 1

gyrostat; the system of coordinates $Gx_1y_1z_1$ has axes of unchanged direction, parallel to the corresponding axes of the system of coordinates $O'xyz$ (see the figure).

We will introduce a system of principal axes of inertia $G\xi\eta\zeta$ half-connected with the gyrostat. The η axis always lies in the plane of the vertical meridian, passing through the point O of contact of the body with the reference plane and the ζ axis, and perpendicular to the latter, while the ξ axis is orthogonal to the $\eta\zeta$ plane. Since the ξ axis is parallel to the nodal line, for the system $G\xi\eta\zeta$ the angle of natural rotation is zero at each instant of time.

The position of the gyrostat in the $O'xyz$ system is defined by the coordinates x_G, y_G of the centre of mass, the Euler angles ϑ, ψ, φ of the body and the angle ζ of rotation of the rotor with respect to the body. The nutation angle ϑ is the angle betweedn the ζ and z_1 axes. We will assume that ϑ, ψ, φ vary within the following limits

$$0 \leqslant \vartheta \leqslant \pi/2, \quad 0 \leqslant \psi < 2\pi, \quad 0 \leqslant \varphi < 2\pi$$

Suppose A and $B = A$ are the moments of inertia of the gyrostat about the ξ, η axes and C and J are the moments of inertia of the body and the rotor respectively about the ζ axis.

In the $G\xi\eta\zeta$ system the point O has coordinated 0, η, ζ. The variables η, ζ can be considered as unique functions of the angle ϑ, defining the curve which bounds the meridian section [14, 11]. The distance from the centre of mass to the plane and its derivative with respect to ϑ have the form [14, 11]

$$h(\vartheta) = -\eta\sin\vartheta - \zeta\cos\vartheta, \quad h'(\vartheta) = -\eta\cos\vartheta + \zeta\sin\vartheta \tag{1.1}$$

From (1.1) we have

$$\eta = -h\sin\vartheta - h'\cos\vartheta, \quad \zeta = -h\cos\vartheta + h'\sin\vartheta \tag{1.2}$$

Differentiating (1.2) with respect to ϑ, we obtain

$$\eta' = -[h(\vartheta) + h''(\vartheta)]\cos\vartheta, \quad \zeta' = [h(\vartheta) + h''(\vartheta)]\sin\vartheta \tag{1.3}$$

Suppose p, q and r are the projections onto the ξ, η, ζ axes of the vector $\boldsymbol{\omega}$ of the instantaneous angular velocity of the body. The Euler kinematic equations have the form

$$p = \dot{\vartheta}, \quad q = \dot{\psi}\sin\vartheta, \quad r = \dot{\psi}\cos\vartheta + \dot{\varphi} \tag{1.4}$$

The projections of the vector of the velocity of the centre of mass G of the body onto the axes of the $O'xyz$ system are found from the condition for the body to roll along the plane without slipping

$$\dot{x}_G = m_{11}\dot{\vartheta} + m_{12}\dot{\varphi} + m_{13}\dot{\psi}, \quad \dot{y}_G = m_{21}\dot{\vartheta} + m_{22}\dot{\varphi} + m_{23}\dot{\psi}$$
$$\dot{z}_G = -(\eta\cos\vartheta - \zeta\sin\vartheta)\dot{\vartheta} \tag{1.5}$$

where

$$m_{11} = -\sin\psi(\eta\sin\vartheta + \zeta\cos\vartheta), \quad m_{21} = \cos\psi(\eta\sin\vartheta + \zeta\cos\vartheta)$$
$$m_{12} = \eta\cos\psi, \quad m_{22} = \eta\sin\psi \tag{1.6}$$
$$m_{13} = \cos\psi(\eta\cos\vartheta - \zeta\sin\vartheta), \quad m_{23} = \sin\psi(\eta\cos\vartheta - \zeta\sin\vartheta)$$

The first two equations of (1.5) are non-integrable kinematic constraints, while the last one is a consequence of the geometrical constraint $z_G = h(\vartheta)$. The equations of the non-holonomic constraints can be represented in the form [13]

$$\dot{\mathbf{y}} = \mathbf{M}\dot{\mathbf{x}} \tag{1.7}$$
$$\mathbf{x} = \{\vartheta, \varphi, \psi\}, \quad \mathbf{y} = \{x_G, y_G\}; \quad \mathbf{M} = \| m_{\varkappa i} \|, \quad \varkappa = 1,2; \quad i = 1,2,3$$

Here $\dot{\mathbf{x}}$ are independent generalized velocities and $\dot{\mathbf{y}}$ are dependent generalized velocities.

The projections of the absolute velocity of the point O onto the $Gx_1y_1z_1$ axes have the form

$$\dot{x}_{1O} = \eta\cos\psi\dot{\varphi} - \sin\psi(\eta'\cos\vartheta - \zeta'\sin\vartheta)\dot{\vartheta}$$
$$\dot{y}_{1O} = \eta\sin\psi\dot{\varphi} + \cos\psi(\eta'\cos\vartheta - \zeta'\sin\vartheta)\dot{\vartheta} \tag{1.8}$$
$$\dot{z}_{1O} = (\eta'\sin\vartheta + \zeta'\cos\vartheta)\dot{\vartheta}$$

We have following expressions for the kinetic energy and potential energy of the gyrostat

$$T = \frac{1}{2}m\left[\dot{\xi}_G^2 + \dot{\eta}_G^2 + \dot{\zeta}_G^2\right] + \frac{1}{2}[A(p^2 + q^2) + Cr^2] + \frac{1}{2}J(\dot{\delta} + r)^2 \tag{1.9}$$
$$V = mgh(\vartheta)$$

The projections of the velocity vector of the centre of mass G onto the axes of the $G\xi\eta\zeta$ system have the form

$$\dot{\xi}_G = \eta(\dot{\psi}\cos\vartheta + \dot{\varphi}) - \zeta\dot{\psi}\sin\vartheta, \quad \dot{\eta}_G = \zeta\dot{\vartheta}, \quad \dot{\zeta}_G = -\eta\dot{\vartheta}$$

We will assume that the generalized forces acting on the rotor are equal to zero. From the equation of motion of the rotor we have the first integral

$$\dot{\delta} + r = \Omega = \text{const} \tag{1.10}$$

which expresses the fact that the projection onto the axis of rotation of the rotor and its instantaneous absolute angular velocity of rotation are constant.

Obviously a heavy gyrostat on an absolutely rough horizontal plane is a non-holonomic Chaplygin system, since T, V and the constraint factors $m_{\varkappa h}$ are independent of the generalized coordinates x_G and y_G [1.11].

2. THE EQUATIONS OF MOTION

We will take the equations of motion of the system in Chaplygin's form [1]

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot{x}_i} - \frac{\partial T_*}{\partial x_i} + \frac{\partial V}{\partial x_i} + \sum_{\varkappa=1}^{2}\Theta_\varkappa\left[\sum_{j=1}^{3}\left(\frac{\partial m_{\varkappa j}}{\partial x_i} - \frac{\partial m_{\varkappa i}}{\partial x_j}\right)\dot{x}_j\right] = 0, \quad i = 1,2,3 \tag{2.1}$$

where

$$T_* = \frac{1}{2}A_1\dot{\vartheta}^2 + \frac{1}{2}B_1\dot{\psi}^2\sin^2\vartheta + \frac{1}{2}C_1(\dot{\psi}\cos\vartheta + \dot{\varphi})^2 -$$
$$-E\dot{\psi}\sin\vartheta(\dot{\psi}\cos\vartheta + \dot{\varphi}) + \frac{1}{2}J(\dot{\delta} + \dot{\psi}\cos\vartheta + \dot{\varphi})^2 \tag{2.2}$$

$$A_1 = A + m(\eta^2 + \zeta^2), \quad B_1 = A + m\zeta^2$$
$$C_1 = C + m\eta^2, \quad E = m\eta\zeta \tag{2.3}$$

Here T_*, $\Theta_\varkappa$ is the result of eliminating the quantities $\dot{y}_\varkappa$ using relations (1.7) from T and $\partial T/\partial\dot{y}_\varkappa$ respectively.

Converting the last term on the left-hand side of the i-th equation of (2.1), these equations can be rewritten in the form

$$\frac{d}{dt}\frac{\partial T_*}{\partial\dot{x}_i} - \frac{\partial T_*}{\partial x_i} + \frac{\partial V}{\partial x_i} - \sum_{j,h=1}^{3}\omega_{ijh}\dot{x}_j\dot{x}_h = 0, \quad i = 1, 2, 3$$

where we have introduced the coefficients of the non-holonomicity terms [13]

$$\omega_{ijh} = \sum_{\varkappa=1}^{2}\left(\frac{\partial m_{\varkappa i}}{\partial x_j} - \frac{\partial m_{\varkappa j}}{\partial x_i}\right)\frac{\partial\Theta_\varkappa}{\partial\dot{x}_h} = -\omega_{jih}, \quad \omega_{iih} = 0$$

which, for the problem considered, are

$$\omega_{121} = \omega_{131} = 0, \quad \omega_{122} = -m\eta\eta', \quad \omega_{123} = -m(\eta\cos\vartheta - \zeta\sin\vartheta)\eta'$$
$$\omega_{132} = -m\eta(\eta'\cos\vartheta - \zeta'\sin\vartheta) \tag{2.4}$$
$$\omega_{133} = -m(\eta\cos\vartheta - \zeta\sin\vartheta)(\eta'\cos\vartheta - \zeta'\sin\vartheta)$$
$$\omega_{231} = m\eta(\eta\sin\vartheta + \zeta\cos\vartheta), \quad \omega_{232} = \omega_{233} = 0$$

Hence, the equations of motion of the gyrostat take the form

$$\frac{d}{dt}[A_1\dot{\vartheta}] - m(\eta\eta' + \zeta\zeta')\dot{\vartheta}^2 + [C_1(\dot{\psi}\cos\vartheta + \dot{\varphi}) + J\Omega - B_1\dot{\psi}\cos\vartheta]\dot{\psi}\sin\vartheta -$$
$$-E[\dot{\psi}^2\sin^2\vartheta - \dot{\psi}\cos\vartheta(\dot{\psi}\cos\vartheta + \dot{\varphi})] = -mgh'(\vartheta)$$
$$\frac{d}{dt}[B_1\dot{\psi}\sin\vartheta - E(\dot{\psi}\cos\vartheta + \dot{\varphi})] + E\dot{\psi}\sin\vartheta\dot{\vartheta} +$$
$$+ [B_1\dot{\psi}\cos\vartheta - C_1(\dot{\psi}\cos\vartheta + \dot{\varphi}) - J\Omega + m\eta^2(\dot{\psi}\cos\vartheta + \dot{\varphi})]\dot{\vartheta} + \tag{2.5}$$
$$+ m\zeta'[\eta(\dot{\psi}\cos\vartheta + \dot{\varphi}) - \zeta\dot{\psi}\sin\vartheta]\dot{\vartheta} = 0$$
$$\frac{d}{dt}[C_1(\dot{\psi}\cos\vartheta + \dot{\varphi}) + J\Omega - E\dot{\psi}\sin\vartheta] - [m\eta^2\dot{\psi}\sin\vartheta + E\dot{\psi}\cos\vartheta]\dot{\vartheta} -$$
$$-m\eta'[\eta(\dot{\psi}\cos\vartheta + \dot{\varphi}) - \zeta\dot{\psi}\sin\vartheta]\dot{\vartheta} = 0$$

When $J\Omega = 0$ Eqs (2.5) take the form of the equations of motion without slipping of a heavy solid of revolution on a fixed horizontal plane [14, 12].

Hence, Chaplygin's equations (2.5) express, in projections onto the axis of a semimobile system of coordinates, the theorem of the angular momentum of a gyrostat about a fixed point O.

3. ROUTH'S FUNCTION

It can be seen from the second expression of (1.9) and relations (2.2) and (2.4) that the kinetic energy T_*, the potential energy V and the non-zero coefficients ω_{ijh} are independent of the generalized coordinates φ and ψ, i.e. the coordinates φ and ψ are pseudocyclic [13].

We will change from generalized velocities to the momenta of pseudocyclic coordinates

$$p_1 = \partial T_* / \partial\dot\varphi = C_1 r - Eq + J\Omega \tag{3.1}$$

$$p_2 = \partial T_* / \partial\dot\psi = [B_1 q - Er]\sin\vartheta + [C_1 r - Eq + J\Omega]\cos\vartheta$$

and we will introduce the analogue of Routh's function

$$R = T_* - V - p_1\dot\varphi - p_2\dot\psi - J\Omega\dot\delta$$

Expressing the variables q and r from (3.1) in terms of p_1 and p_2 and neglecting the unimportant additive constant, we obtain the following relation for Routh's function

$$R = R_2 - W = \frac{1}{2}A_1\dot\vartheta^2 - \frac{1}{2}\alpha_1(p_1 - J\Omega)^2 - \frac{1}{2}\alpha_2(p_2 - p_1\cos\vartheta)^2 -$$
$$-\alpha_3(p_1 - J\Omega)(p_2 - p_1\cos\vartheta) - mgh(\vartheta)$$

$$R_2 = \frac{1}{2}A_1\dot\vartheta^2, \quad W = mgh(\vartheta) + \frac{1}{2}\alpha_1(p_1 - J\Omega)^2 +$$
$$+\frac{1}{2}\alpha_2(p_2 - p_1\cos\vartheta)^2 + \alpha_3(p_1 - J\Omega)(p_2 - p_1\cos\vartheta)$$

$$\alpha_1 = \frac{B_1}{\Delta}, \quad \alpha_2 = \frac{C_1}{\Delta\sin^2\vartheta}, \quad \alpha_3 = \frac{E}{\Delta\sin\vartheta}, \quad \Delta = B_1C_1 - E^2$$

In the new variables, Eqs (2.4) take the form

$$\frac{d}{dt}\frac{\partial R_2}{\partial\dot\vartheta} = \frac{\partial R_2}{\partial\vartheta} - \frac{DW}{D\vartheta}, \quad \dot{\mathbf{p}} = \mathbf{\Gamma}\dot\vartheta \tag{3.2}$$

Here

$$\frac{D}{D\vartheta} = \frac{\partial}{\partial\vartheta} + \mathbf{\Gamma}^T\frac{\partial}{\partial\mathbf{p}}, \quad \mathbf{\Gamma} = \begin{Vmatrix}\gamma_{21} \\ \gamma_{22}\end{Vmatrix}$$

$$\gamma_{l1} = (\omega_{l12} + \omega_{l21})\frac{\partial W}{\partial p_1} + (\omega_{l13} + \omega_{l31})\frac{\partial W}{\partial p_2}, \quad l = 2,3$$

The equations of motion admit of the energy integral

$$T_* + V = R_2 + W = \text{const}$$

and two first integrals of the form $\mathbf{U}(\vartheta, \mathbf{p}) = \mathbf{c}$, explicit expressions for which are not known [13].

4. STEADY MOTIONS

System (3.2) has the steady solutions [13]

$$\vartheta = \vartheta_0, \quad \dot\vartheta = 0, \quad p_1 = P_1, \quad p_2 = P_2 \tag{4.1}$$

where the constants ϑ_0, P_1 and P_2 satisfy the equation

$$DW / D\vartheta = 0 \tag{4.2}$$

Hence, the steady motions form a two-parameter family, defined by the relation

$$mgh'(\vartheta_0) + \beta_1(P_1 - J\Omega)(P_1 - P_2\cos\vartheta_0) + \beta_2(P_1 - P_2\cos\vartheta_0)(P_2 - P_1\cos\vartheta_0) = 0 \tag{4.3}$$

$$\beta_1 = \frac{E_0}{\Delta_0\sin^2\vartheta_0}, \quad \beta_2 = \frac{C_1^0}{\Delta_0\sin^3\vartheta_0}, \quad \Delta_0 = (B_1C_1 - E^2)\big|_{\vartheta=\vartheta_0}$$

We will consider some special types of steady motions.
Permanent rotations around an axis of symmetry

$$\vartheta_0 = 0, \quad \dot{\vartheta} = 0, \quad P_1 = P_2 = P \tag{4.4}$$

$$(\dot{\varphi} + \dot{\psi} = \dot{\varphi}_0 + \dot{\psi}_0 = r_0)$$

These occur when

$$h'(0) = 0$$

It follows from relations (1.5) and (1.8), when (4.4) holds, that the centre of mass G and the point of contact O of the gyrostat are fixed in these motions, which represent rotation around a fixed vertical axis of symmetry with an arbitrary constant angular velocity r_0.

Permanent rotations: the general case

$$\vartheta_0 \neq 0, \quad \dot{\vartheta} = 0, \quad P_2 = P_1 \cos\vartheta_0 + \frac{b_1}{b_2} \sin\vartheta_0 (P_1 - J\Omega) \tag{4.5}$$

$$(\dot{\varphi} = 0, \quad \dot{\psi}_0 \neq 0)$$

Here

$$b_1 = B_1^0 \sin\vartheta_0 - E_0 \cos\vartheta_0, \quad b_2 = C_1^0 \cos\vartheta_0 - E_0 \sin\vartheta_0$$

where the constant P_1 satisfies the equation

$$mgh'(\vartheta_0) + \frac{\sin\vartheta_0}{b_2} P_1(P_1 - J\Omega) - \frac{b_1 \cos\vartheta_0}{b_2^2}(P_1 - J\Omega)^2 = 0$$

which, after changing to the generalized velocity $\dot{\psi}_0$, is converted to the form

$$mgh'(\vartheta_0) = \Lambda(\vartheta_0) - J\Omega\dot{\psi}_0 \sin\vartheta_0$$

$$\Lambda(\vartheta_0) = [(B_1^0 - C_1^0)\sin\vartheta_0 \cos\vartheta_0 + E_0(\sin^2\vartheta_0 - \cos^2\vartheta_0)]\dot{\psi}_0^2$$

The direction cosines of kinematically possible axes of permanent rotations of the gyrostat must satisfy this equation. When there is no rotor we have

$$mgh'(\vartheta_0) = \Lambda(\vartheta_0)$$

However, the only axes which are dynamically permissible will be those whose direction cosines satisfy the inequality

$$(J\Omega)^2 \sin^2\vartheta_0 + 4\Lambda(\vartheta_0) mgh'(\vartheta_0) \geqslant 0 \tag{4.6}$$

If there is no rotor, the inequality reduces to the form

$$\Lambda(\vartheta_0) mgh'(\vartheta_0) \geqslant 0 \tag{4.7}$$

Inequalities (4.6) and (4.7) show that the rotating rotor connected to the body widens the set of dynamically permissible axes of permanent rotations.

Solution (4.5) corresponds to the steady motion of a rigid body in which the body rests on horizontal plane at one point of its surface and rotates with angular velocity $\dot{\psi}_0$ around the vertical l passing through this point. The centre of gravity G of the gyrostat then describes a circle in the plane $z = h(\vartheta_0)$, parallel to the reference plane, with centre on the axis l and radius

$$\rho_G = |\mathbf{v}_G| / |\dot{\psi}_0| = |\eta_0 \cos\vartheta_0 - \zeta_0 \sin\vartheta_0|$$

Rectilinear rolling

$$\vartheta_0 \neq 0, \quad \dot{\vartheta} = 0, \quad P_2 = P_1 \cos\vartheta_0 - \frac{E_0}{C_1^0} \sin\vartheta_0 (P_1 - J\Omega)$$

$$(\dot{\psi} = 0, \quad \dot{\varphi}_0 \neq 0) \tag{4.8}$$

where P_1 satisfies the equation

$$mgh'(\vartheta_0) + (P_1 - J\Omega)[P_1 \sin\vartheta_0 + \frac{E_0}{C_1^0} \cos\vartheta_0 (P_1 - J\Omega)] \left[\beta_1 - \beta_2 \frac{E_0}{C_1^0} \sin\vartheta_0\right] \sin\vartheta_0 = 0$$

Substituting the expression $P_1 = C_1\dot{\varphi}_0 + J\Omega$ into this equation we obtain

$$h'(\vartheta_0) = 0 \tag{4.9}$$

It follows from (1.5), (1.8) and (4.8) that steady motions of the gyrostat for which the point O describes a straight line while the constant angular velocity $\dot{\varphi}_0$ of the rolling is arbitrary correspond to relations (4.8).

Motion of the regular precession type

$$\vartheta_0 \neq 0, \quad \dot{\vartheta} = 0, \quad p_1 = P_1, \quad p_2 = P_2 \tag{4.10}$$

$$(\dot{\varphi} = \dot{\varphi}_0 \neq 0, \quad \dot{\psi} = \dot{\psi}_0 \neq 0)$$

The constants ϑ_0, P_1 and P_2 satisfy Eq. (4.3).

In this motion the gyrostat rotates with constant angular velocity $\dot{\varphi}_0$ around its own axis of rotation ζ, and $\dot{\psi}_0$ about the vertical. In this case the centre of mass G of the gyrostat describes a circle of radius ρ_G in a plane parallel to the reference plane, while the point of contact O of the body with the plane describes a circle of radius ρ_O, where

$$\rho_G = \left|\frac{\mathbf{v}_G}{\dot{\psi}_0}\right| = \left|\frac{\eta_0\dot{\varphi}_0 + (\eta_0 \cos\vartheta_0 - \zeta_0 \sin\vartheta_0)\dot{\psi}_0}{\dot{\psi}_0}\right|$$

$$\rho_O = \left|\frac{\mathbf{v}_O}{\dot{\psi}_0}\right| = \left|\frac{\eta_0\dot{\varphi}_0}{\dot{\psi}_0}\right|$$

If we digress from the motion of the point of contact O about the reference plane and concentrate out attention only on the orientation of the system $\xi\eta\zeta$ with respect to the fixed system, the motion reduces solely to regular precession about a vertical axis [14].

5. THE STABILITY OF THE STEADY MOTIONS

As we know [13, pp. 100, 101 and 129], the steady motion (4.1) is stable if all the eigenvalues of the matrix

$$\mathbf{C} = \| D^2W / D\vartheta^2 \|$$

are positive at the point (ϑ_0, $\mathbf{p}_0$), and unstable if the determinant of the matrix C is less than zero at the point (ϑ_0, $\mathbf{p}_0$).

In the case considered, the sufficient condition for the steady motions of the gyrostat to be stable reduces to the inequality

$$\frac{D^2W}{D\vartheta^2} = \frac{\partial}{\partial\vartheta}\frac{DW}{D\vartheta} + \mathbf{\Gamma}^T \frac{\partial}{\partial\mathbf{p}}\frac{DW}{D\vartheta} > 0$$

The steady motions are unstable if this inequality strictly breaks down.

We will obtain the necessary and sufficient condition for the steady motions of the gyrostat to be stable for any angle $\vartheta_0 \neq 0$

$$L_1 q_0^2 + L_2 q_0 r_0 + L_3 r_0^2 + L_4 J\Omega q_0 + L_5 J\Omega r_0 + \frac{C_1^0}{\Delta_0}(J\Omega)^2 + mgh''(\vartheta_0) > 0 \tag{5.1}$$

where

$$L_1 = 3E_0 \operatorname{ctg}\vartheta_0 + 2B_1^0 \operatorname{ctg}^2\vartheta_0 + m\eta_0^2 + \frac{B_1^0}{\sin^2\vartheta_0} + \frac{C}{\Delta_0}\tilde{B}\operatorname{ctg}\vartheta_0 m\zeta_0[\eta_0 + \zeta_0']$$

$$L_2 = -\frac{E_0}{\sin^2\vartheta_0} - \frac{C}{\Delta_0}m\zeta_0\zeta_0'[C_1^0 - B_1^0\operatorname{ctg}^2\vartheta_0] - 2\frac{CC_1^0}{\Delta_0}\tilde{B}\operatorname{ctg}\vartheta_0 + \tilde{C}\operatorname{ctg}\vartheta_0$$

$$L_3 = \frac{C\tilde{C}}{\Delta_0}[C_1^0 - m\zeta_0\zeta_0'\operatorname{ctg}\vartheta_0] \tag{5.2}$$

$$L_4 = -2\operatorname{ctg}\vartheta_0 - \frac{C_1^0}{\Delta_0}\tilde{B}\operatorname{ctg}\vartheta_0 - \frac{Cm\zeta_0}{\Delta_0}[\eta_0 + \zeta_0']$$

$$L_5 = \frac{C_1^0\tilde{C}}{\Delta_0} + \frac{C}{\Delta_0}[C_1^0 - m\zeta_0\zeta_0'\operatorname{ctg}\vartheta_0]$$

$$\tilde{B} = A - m\zeta_0\frac{h(\vartheta_0)}{\cos\vartheta_0}, \qquad \tilde{C} = C - m\eta_0\frac{h(\vartheta_0)}{\sin\vartheta_0} \tag{5.3}$$

When $J\Omega = 0$ inequality (5.1) is the sufficient condition for the steady motions (4.1) of the solid of revolution to be stable, assuming that $\vartheta_0 \neq 0$ [12].

Consider steady motion of the form

$$\vartheta_0 \neq 0, \quad \dot{\vartheta} = 0, \quad q_0 = 0, \quad r_0 \neq 0 \tag{5.4}$$

This motion is possible if condition (4.9) is satisfied.

We obtain the sufficient condition for steady motions (4.3) to be stable from inequality (5.1)

$$\frac{D^2W}{D\vartheta^2} = \frac{1}{\Delta_0}[\tilde{L}_3 r_0^2 + \tilde{L}_5 J\Omega r_0 + C_1^0(J\Omega)^2 + mgh''(\vartheta_0)\Delta_0] > 0 \tag{5.5}$$

where we can write the coefficients $\tilde{L}_3, \tilde{L}_5$ in the form

$$\tilde{L}_3 = C[C_1^0 + m\eta_0^2\operatorname{ctg}^2\vartheta_0]\left[C_1^0 + mh(\vartheta_0)\frac{\cos^2\vartheta_0}{\sin\vartheta_0}\zeta_0'\right]$$

$$\tilde{L}_5 = C_1^0\tilde{C} + C[C_1^0 - m\zeta_0\zeta_0'\operatorname{ctg}\vartheta_0]$$

Note that in the case considered the curvature k of the curve bounding the meridian section at the point of contact of the body with the plane is positive. Since [15]

$$k(\vartheta_0) = -1/[h(\vartheta_0) + h''(\vartheta_0)]$$

from the inequality $k(\vartheta_0) > 0$ we obtain the condition

$$h''(\vartheta_0) < -h(\vartheta_0) < 0 \tag{5.6}$$

Assertion. 1. If $\tilde{L}_3 > 0, h''(\vartheta_0)$, steady motion (5.4) is stable when $r_0 \in (-\infty, r_{01}) \cup (r_{02}, +\infty)$ and unstable when $r_0 \in (r_{01}, r_{02})$.

2. If $\tilde{L}_3 < 0$ and the quantity $\Omega \in (-\infty, -\sqrt{K_*}/J) \cup (\sqrt{K_*}/J, +\infty)$, steady motion (5.4) is stable when $r_0 \in (r_{01}, r_{02})$ and unstable when $r_0 \in (-\infty, r_{01}) \cup (r_{02}, +\infty)$. If $\Omega \in [-\sqrt{K_*}/J, \sqrt{K_*}/J$, steady motion (5.4) is unstable for any value of r_0.

Bearing in the mind the inequality

$$\Delta_0 = AC + Am\eta_0^2 + Cm\zeta_0^2 > 0 \tag{5.7}$$

we will investigate the sign of the quadratic function on the left-hand side of inequality (5.5), for various values of the constants Ω and r_0. The discriminant of this quadratic function is

$$d = [\tilde{L}_5^2 - 4\tilde{L}_3 C_1^0](J\Omega)^2 - 4\tilde{L}_3 mgh''(\vartheta_0)\Delta_0$$

It can be shown that

$$\tilde{L}_5^2 - 4\tilde{L}_3 C_1^0 = m^2[C_1^0\eta_0 h(\vartheta_0) + C\zeta_0\zeta_0' \cos\vartheta_0]^2 / \sin^2\vartheta_0 = \chi^2 \geqslant 0$$

where the equality holds if

$$\zeta_0' = -C_1\eta_0 h(\vartheta_0)/(C\zeta_0 \cos\vartheta_0) \tag{5.8}$$

Suppose case 1 occurs. then the discriminant d of the quadratic function $D^2W/D\vartheta^2$ is positive for any value of the absolute instantaneous angular velocity Ω of rotation of the rotor. In this case real different $(r_0)_{1,2} = (-\tilde{L}_5 J\Omega \mp \sqrt{d})/2\tilde{L}_3$ exist such that when $r_0 \in (-\infty, r_{01}) \cup (r_{02}, +\infty)$ the steady motion (5.4) is stable.

Note that the discriminant d takes the minimum value when $\Omega = 0$, i.e. when the rotor does not rotate or is not present, or if equality (5.8) holds. In this case

$$(r_0)_{1,2} = \mp\sqrt{-mgh''(\vartheta_0)\Delta_0 / \tilde{L}_3}$$

If case 2 occurs, then $h''(\vartheta_0) < 0$. The discriminant d can then take positive values only if

$$\Omega \in (-\infty, -\sqrt{K_*}/J) \cup (\sqrt{K_*}/J, +\infty)$$
$$K_* = 4\tilde{L}_3 mgh''(\vartheta_0)\Delta_0/\chi^2, \qquad \chi^2 \neq 0$$

In this case the leading coefficient $\tilde{L}_3$ of the quadratic function is negative and the discriminant d is positive. Consequently, inequality (5.5) holds when $r_0 \in (r_{01}, r_{02})$. For relatively small values of the angular velocity Ω of rotation of the rotor

$$\Omega \in [-\sqrt{K_*}/J, \ \sqrt{K_*}/J] \tag{5.9}$$

or, when equality (5.8) is satisfied, the discriminant d takes negative values or is equal to zero, and consequently, the quadratic function is negative for any r_0. This means that steady motion (5.4) is unstable. The section (5.9) includes the zero value of Ω, which occurs when the rotor does not rotate or is not present. the instability in this case was pointed out earlier [12].

We will now consider steady motions of the form

$$\vartheta_0 \neq 0, \quad \dot{\vartheta} = 0, \quad q_0 \neq 0, \quad r_0 \neq 0 \tag{5.10}$$

We will write the sufficient condition for stability in the form

$$\frac{D^2W}{D\vartheta^2} = F_1 r_0^2 + F_2 q_0 r_0 + L_1 q_0^2 + mgh''(\vartheta_0) > 0$$

$$F_1 = L_3 + L_5 J\frac{\Omega}{r_0} + \frac{C_1^0}{\Delta_0} J^2 \left(\frac{\Omega}{r_0}\right)^2, \qquad F_2 = L_2 + L_4 J\frac{\Omega}{r_0}$$

If the inequalities

$$F_1 > 0, \quad F_1 L_1 - \frac{1}{4}F_2^2 > 0$$

are satisfied and $h''(\vartheta_0) > 0$, steady motion (5.10) is stable. If $h''(\vartheta_0) < 0$, stability and instability domains may exist.

We will now consider special cases of the steady motion of the gyrostat distinguished above.

Permanent rotation around an axis of symmetry. We put $p_1 = p_2 = P$ in the expression for $DW/D\vartheta$ [16]. Then

$$\frac{DW}{D\vartheta} = mgh'(\vartheta) + \frac{E}{\Delta}\frac{P(P-J\Omega)}{1+\cos\vartheta} + \frac{C_1}{\Delta}\frac{P^2\sin\vartheta}{(1+\cos\vartheta)^2}$$

Differentiating this expression, assuming $\Omega = 0$ and using the equation $P = Cr_0 + J\Omega$, we obtain the necessary and sufficient condition for stability

$$[Cr_0 + J\Omega]^2 + 2[Cr_0 + J\Omega]mh(0)[h(0)+h''(0)]r_0 + 4[A + mh^2(0)]mgh''(0) > 0$$

Permanent rotations: the general case. The necessary and sufficient condition for the stability of permanent rotations can be written in the form

$$\frac{1}{\Delta_0}[\bar{L}_1\dot{\psi}_0^2 + \bar{L}_2 J\Omega\dot{\psi}_0 + C_1^0(J\Omega)^2 + mgh''(\vartheta_0)\Delta_0] > 0 \tag{5.11}$$

$$\bar{L}_1 = L_1\sin^2\vartheta_0 + L_2\sin\vartheta_0\cos\vartheta_0 + L_3\cos^2\vartheta_0$$

$$\bar{L}_2 = L_4\sin\vartheta_0 + L_5\cos\vartheta_0$$

We will investigate the quadratic trinomial on the left-hand side of inequality (5.11). We will write its discriminant in the form

$$d = a_*(J\Omega)^2 - 4mgh''(\vartheta_0)\Delta_0\bar{L}_1, \quad a_* = \bar{L}_2^2 - 4\bar{L}_1 C_1^0$$

Bearing inequalities (5.6) and (5.7) in mind, we have the following possible cases. If the parameters of the system are such that $\bar{L}_1 > 0$ and $a_* > 0$, then for any value of Ω real different $\dot{\psi}_{01}$, $\dot{\psi}_{02}$, $(\dot{\psi}_0)_{1,2} = (-\bar{L}_2 J\Omega \mp \sqrt{d})/2\bar{L}_1)$ exist such that permanent rotations of the gyrostat are stable when $\dot{\psi}_0 \in (-\infty, \dot{\psi}_{01}) \cup (\dot{\psi}_{02}, +\infty)$ and unstable when $\dot{\psi}_0 \in (\dot{\psi}_{01}, \dot{\psi}_{02})$. The instability domain is narrower, the smaller the value of Ω.

If the inequalities $\bar{L}_1 > 0$, $a_* < 0$ hold, then when

$$\Omega^2 < \Omega_*^2, \quad \Omega_*^2 = 4|\bar{L}_1|mg|h''(\vartheta_0)|\Delta_0/(|a_*|J^2)$$

permanent rotations are stable when $\dot{\psi}_0 \in (-\infty, \dot{\psi}_{01}) \cup (\dot{\psi}_{02}, +\infty)$. If

$$\Omega^2 > \Omega_*^2$$

permanent rotations are stable for any value of $\dot{\psi}_0$. This means that, by increasing the absolute instantaneous angular velocity Ω of rotation of the rotor, it is possible to stabilize the permanent rotations of the gyrostat.

When the inequalities $\bar{L}_1 < 0$, $a_* > 0$ hold, then, if

$$\Omega^2 > \Omega_*^2$$

permanent rotations are stable when $\dot{\psi}_0 \in (\dot{\psi}_{01}, \dot{\psi}_{02})$. If

$$\Omega^2 < \Omega_*^2$$

permanent rotations are unstable for any value of $\dot{\psi}_0$. In this case, when the modulus of the angular velocity Ω increases the domain of stability of permanent rotations is widened.

Suppose, finally, that $\bar{L}_1 < 0$, $a_* > 0$; then permanent rotations are unstable for any values of Ω and $\dot{\psi}_0$.

Rectilinear rolling. The necessary and sufficient condition for stability has the form

$$mgh''(\vartheta_0) + [\tilde{C}\dot{\varphi}_0 + J\Omega]\left[\frac{C(C + m\eta_0^2 + m\eta_0'\zeta_0)}{\Delta_0}\dot{\varphi}_0 + \frac{C + m\eta_0^2}{\Delta_0}J\Omega\right] > 0$$

This condition, apart from the notation and taking into account corrected misprints, is identical with the result obtained earlier in [4]. The assertion formulated above holds for this motion.

Motion of the regular-precession type. The necessary and sufficient condition for stability of this motion is given by inequality (5.1).

6. A GYROSTAT WITH A CIRCULAR BASE

Suppose the body of the gyrostat is a symmetrical body whose base is a circular disc with a sharp edge, and the plane of the base is perpendicular to the axis of symmetry of the body. Also suppose contact between the gyrostat and the reference plane only occurs at points of the disc contour. Then [14]

$$\eta = -r = \text{const}, \qquad \zeta = -a = \text{const} \tag{6.1}$$

Formulae (1.1) and (1.2) give

$$h = r\sin\vartheta + a\cos\vartheta, \qquad h' = r\cos\vartheta - a\sin\vartheta \tag{6.2}$$

The moments of inertia became constant quantities

$$A_1 = A + m(r^2 + a^2), \qquad B_1 = A + ma^2, \qquad C_1 = C + mr^2, \qquad E = mra$$

The equation for steady motions takes the form

$$[B_1 \operatorname{ctg}\vartheta_0 + E]q_0^2 - [(C_1 + E\operatorname{ctg}\vartheta_0)r_0 + J\Omega]q_0 - mg(r\cos\vartheta_0 - a\sin\vartheta_0) = 0$$

The necessary and sufficient condition for the stability of steady motion (4.1) is obtained from inequality (5.1)

$$\hat{L}_1 q_0^2 + \hat{L}_2 q_0 r_0 + \frac{\hat{L}_3}{\Delta} r_0^2 + \hat{L}_4 J\Omega q_0 + \frac{\hat{L}_5}{\Delta} J\Omega r_0 + \frac{C_1}{\Delta}(J\Omega)^2 + mgh''(\vartheta_0) > 0$$

where the coefficients $\hat{L}_i$ are obtained from L_i taking Eqs (6.1) into account

$$\hat{L}_1 = 3E\operatorname{ctg}\vartheta_0 + 2B_1\operatorname{ctg}^2\vartheta_0 + mr^2 + \frac{B_1}{\sin^2\vartheta_0} + \frac{CE}{\Delta}\tilde{B}\operatorname{ctg}\vartheta_0$$

$$\hat{L}_2 = -\frac{E}{\sin^2\vartheta_0} - 2\frac{CC_1}{\Delta}\tilde{B}\operatorname{ctg}\vartheta_0 + \tilde{C}\operatorname{ctg}\vartheta_0$$

$$\hat{L}_3 = C\tilde{C}C_1, \qquad \hat{L}_4 = -2\operatorname{ctg}\vartheta_0 - \frac{C_1}{\Delta}\tilde{B}\operatorname{ctg}\vartheta_0 - \frac{CE}{\Delta}$$

$$\hat{L}_5 = (C + \tilde{C})C_1$$

We will consider the rectilinear rolling of such a gyrostat along a plane [14] in more details. We obtain from condition (4.9) and the second equation of (6.2)

$$a = r\operatorname{ctg}\vartheta_0 \tag{6.3}$$

Substituting this relation into the second equation of (5.3) we obtain that $\tilde{C} = C_1 + ma^2$. Then the necessary and sufficient condition for the stability of rectilinear rolling is expressed by the inequality

$$\frac{\hat{L}_3}{\Delta}\dot{\varphi}_0^2 + \frac{\hat{L}_5}{\Delta}J\Omega\dot{\varphi}_0 + \frac{C_1}{\Delta}(J\Omega)^2 + mgh''(\vartheta_0) > 0 \tag{6.4}$$

The coefficients $\hat{L}_3 > 0, \hat{L}_5 > 0$.

Note that

$$h''(\vartheta_0) = -r\sin\vartheta_0 - a\cos\vartheta_0 = -h(\vartheta_0) < 0$$

Consequently, the discriminant of the quadratic function on the left-hand side of inequality (6.4) is

$$d = [C_1 m(r^2 + a^2)]^2 (J\Omega)^2 - 4\hat{L}_3 mgh''(\vartheta_0)\Delta > 0$$

for any value of Ω. Here, the greater the value of Ω, the greater the value of the discriminant. Hence, the rolling of a heavy gyrostat with a circular base along a plane is stable if the angular velocity of the rolling lies in the range

$$\dot{\varphi}_0 \in (-\infty, \dot{\varphi}_{01}) \cup (\dot{\varphi}_{02}, +\infty)$$
$$(\dot{\varphi}_0)_{1,2} = (-\hat{L}_5 J\Omega \mp \sqrt{d})/(2\hat{L}_3)$$

and is unstable if $\dot{\psi}_0 \in (\dot{\psi}_{01}, \dot{\psi}_{02})$. The region of instability domain is narrower, the smaller the value of Ω.

The necessary and sufficient condition for the stability of a body with a circular base rolling along a plane (when there is not rotor) [14, p. 204] has the form

$$\frac{\hat{L}_3}{\Delta}\dot{\varphi}_0^2 + mgh''(\vartheta_0) > 0$$

The limit of the stability domain in this case is given by the equation

$$(\dot{\varphi}_0)_{1,2} = \mp\sqrt{mg\,|\,h''(\vartheta_0)\,|\,\Delta / \hat{L}_3}$$

Hence, the stability domain is widened if the body of the gyrostat and the rotor rotate in the same direction, and is narrowed if the rotations of the body and the rotor occur in opposite directions.

7. A DISC WITH A ROTOR

A special case of a gyrostat with a circular base is a heavy uniform circular disc of mass m and radius r. We will direct the ζ axis to be orthogonal to the plane of the disc. Suppose a rotor, whose axis is perpendicular to the plane of the disc, is connected to the disc without friction. The centre of mass G of the whole system, generally speaking, does not lie in the plane of the disc. In the general case, the results obtained in Section 6 will hold for this gyrostat.

Suppose the following condition [1] holds

$$C_1 r_0 + J\Omega = 0 \tag{7.1}$$

The equation which ϑ_0 and q_0 must satisfy in this case has the form

$$[B_1 \operatorname{ctg}\vartheta_0 + E]q_0^2 + \frac{E}{C_1} J\Omega q_0 \operatorname{ctg}\vartheta_0 - mg(r\cos\vartheta_0 - a\sin\vartheta_0) = 0$$

The necessary and sufficient condition for the stability of steady motions of the gyrostat, with condition (7.1), has the form

$$\frac{D^2 W}{D\vartheta^2} = \frac{q_0^2}{\Delta}[\tilde{B}\operatorname{ctg}\vartheta_0(3\Delta\operatorname{ctg}\vartheta_0 + C_1 E) + B_1\Delta + mr^2(\Delta - \tilde{B}E\operatorname{ctg}\vartheta_0)] +$$
$$+ \frac{J\Omega q_0}{\Delta}\left[E^2\operatorname{ctg}\vartheta_0 + \frac{E}{C_1}\Delta(2\operatorname{ctg}\vartheta_0 + 1) - mr^2(B_1 + \tilde{B})\operatorname{ctg}\vartheta_0\right] -$$
$$- \frac{Emr^2}{C_1\Delta}(J\Omega)^2\operatorname{ctg}\vartheta_0 - mg(r\sin\vartheta_0 + a\cos\vartheta_0) > 0$$

Suppose, when condition (7.1) is satisfied, the gyrostat rolls along a straight line. We obtain

$$\left.\frac{D^2 W}{D\vartheta^2}\right|_{\dot{\varphi}=\varphi_0,\, \dot{\psi}=0} = -\frac{mgr}{\sin\vartheta_0} - \frac{m^2 r^4}{C_1\Delta}(J\Omega)^2\operatorname{ctg}^2\vartheta_0 < 0 \tag{7.2}$$

i.e. the rolling of the gyrostat along a straight line, with condition (7.1), is unstable.

We will investigate the stability of the equilibrium of the gyrostat, which is possible if $h' = 0$, i.e. the plane of the disc is inclined to the reference plane at an angle $\vartheta_0 = \operatorname{arcctg}(a/r)$. Then, bearing the relations $a = r \operatorname{ctg} \vartheta_0$ in mind, we obtain the necessary and sufficient condition for the stability of the equilibrium of the gyrostat

$$\Omega^2 > \frac{mgr}{\sin\vartheta_0}\frac{\Delta}{C_1 J^2} \tag{7.3}$$

If the centre of mass of the disc-rotor system lies in the plane of the disc and coincides with its geometric centre, we have

$$\eta = -r, \qquad \zeta = 0 \tag{7.4}$$

Then the steady motion of the gyrostat will be uniform rolling along the straight line for which the plane of the disc is vertical, or equilibrium of the gyrostat, which is possible when $\vartheta_0 = \pi/2$. Uniform spinning of the gyrostat around the vertically situated diameter of the disc is only possible if $J\Omega = 0$, i.e. when there is no rotor.

The necessary and sufficient condition for the stability of uniform rolling of such a gyrostat has the form [2]

$$[C\dot{\varphi}_0 + J\Omega]^2 + mr^2[C\dot{\varphi}_0 + J\Omega]\dot{\varphi}_0 - Amgr > 0$$

The conditions for the stability of rolling and spinning respectively for a disc without a rotor are defined by the equations [17, 18, 13]

$$C[C + mr^2]\dot{\varphi}_0^2 - Amgr > 0, \qquad [A + mr^2]\dot{\varphi}_0^2 - mgr > 0$$

The necessary and sufficient condition for equilibrium of the gyrostat (7.4) has the form

$$D^2W / D\vartheta^2 = -Amgr + (J\Omega)^2 > 0 \tag{7.5}$$

If the rotor does not rotate or is not present, we have

$$D^2W / D\vartheta^2 = -mgr < 0 \tag{7.6}$$

i.e. the equilibrium position is unstable.

Hence, it follows from inequalities (7.3), (7.5) and (7.6) that a rotating rotor stabilizes the equilibrium of the gyrostat.

For rolling of the gyrostat (7.4) along a straight line in the case of condition (7.1) we obtain

$$D^2W / D\vartheta^2 = -mgr < 0$$

i.e. we have instability.

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